Tutorial Notes Information

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- Prerequisite Knowledge:
  - Differentiation of Vector Valued Functions, Multivariable Differentiation
  - Inner Product & Cross Product
  - Differentiation By Part
- Table of Content & Outline
  - From Curve Theory to Surface Theory
  - Geometric Significance of First Fundamental Form
  - Tangent Space & Calculations
  - Some Exercises
  - Test 2 Revision
- References:
  - **Lecture Notes** of Dr. LAU & Dr. CHENG
- All the suggestions and feedback are welcome. Any report of typos is appreciated.

1 From Curve Theory to Surface Theory

Definition 1.1. A regular parametrised surface is a differentiable function $\mathbf{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $D \subset \mathbb{R}^2$ is an open connected subset such that $\mathbf{X}_u \times \mathbf{X}_v \neq \mathbf{0}$ for any $(u, v) \in U \subset \mathbb{R}^2$. The image $S = \mathbf{X}(D) \subset \mathbb{R}^3$ is called a regular surface

Key Observation. Requiring $\mathbf{X}_u \times \mathbf{X}_v \neq \mathbf{0}$ is equivalent to $\underbrace{\text{global existence of normal vector}}_{\text{non-zero everywhere}}$

Key Question. Any connection between "definition of regular curve" ?
In what sense "regular surface" can be regarded as an analogue of "regular curve" ?
Why "First Fundamental Form of $\mathbf{X}$" contains information of surface S ?

Theorem 1.1. Suppose $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ is a regular space curve, then $[\mathbf{r}'(t)]^T[\mathbf{r}'(t)] \neq 0$

Theorem 1.2. Suppose $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ is a regular surface, then $I = [D\,\mathbf{X}(u, v)]^T[D\,\mathbf{X}(u, v)]$ is invertible. ($\det(I) \neq 0$)

Warm Reminder. It may clarify your doubts in notations

- For $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, the 3×1 vector $\mathbf{r}'(t)$ has enough space to contain all derivative information
- For $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$, we need a 3×2 matrix $D\,\mathbf{X}(u, v)$ to contain all derivative information

Theorem 1.3. Suppose $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ is a regular surface, denote its first fundamental form by I, then

$$I = [D\,\mathbf{X}(u, v)]^T[D\,\mathbf{X}(u, v)]$$
$$\det(I) = \|\mathbf{X}_u \times \mathbf{X}_v\|^2$$

Proof. By pure computation without any intuition. I would say it is an strange coincidence □

Summary & Logic Flow.

$$\begin{array}{ccc}
\text{Curve} & \longrightarrow & \text{Surface} \\
\Big\downarrow{\scriptstyle\text{derivative info}} & & \Big\downarrow{\scriptstyle\text{derivative info}} \\
[\mathbf{r}'(t)]^T[\mathbf{r}'(t)] & \longrightarrow & I = [D\mathbf{X}(u,v)]^T[D\mathbf{X}(u,v)] \\
\Big\downarrow{\scriptstyle\text{det}} & & \Big\downarrow{\scriptstyle\text{det}} \\
\|\mathbf{r}'(t)\|^2 & \longrightarrow & \|\mathbf{X}_u \times \mathbf{X}_v\|^2 \\
\Big\downarrow{\scriptstyle\text{if regular}} & & \Big\downarrow{\scriptstyle\text{if regular}} \\
\underbrace{\mathbf{r}'(t) \neq \mathbf{0}}_{\text{tangent exists}} & \longrightarrow & \underbrace{\mathbf{X}_u \times \mathbf{X}_v \neq \mathbf{0}}_{\text{normal exists}}
\end{array}$$

2 Geometric Significance of First Fundamental Form

Theorem 2.1. Let S be a regular parametrised surface with parametrisation $\mathbf{X}(u,v), (u,v) \in D \subset \mathbb{R}^2$. Denote I as the first fundamental form I with respect to $\mathbf{X}$, then

$$\text{Surface area of } S = \iint_S 1dA = \iint_U \sqrt{\det(I)}dudv$$

Remark. We call "$dA = \sqrt{\det(I)}dudv$" the area element on S

Proof. Lecture Notes Definition 3.2.3. Heuristic Area Distortion Approximisation ☐

Theorem 2.2 (Lecture Notes Chapter 3 Exercise 5)**.** Let $\mathbf{X}(u,v)$ be a regular prarmetrised surface. Let $\mathbf{r}(t) = \mathbf{X}(u(t), v(t))$ be a curve lying on the surface ($a < t < b$), then

$$\text{Arc-length of } \mathbf{r} = \int_C 1ds = \int_a^b \sqrt{\begin{pmatrix}\dot{u} & \dot{v}\end{pmatrix} I \begin{pmatrix}\dot{u} \\ \dot{v}\end{pmatrix}}dudv$$

Proof. Here is an outline of the proof. Details are left as exercise:

1. By multivariable chain rule: $\mathbf{r}'(t) = \mathbf{X}_u\, u'(t) + \mathbf{X}_v\, v'(t)$
2. Compute $\|\mathbf{r}'(t)\|$ by inner product
3. Expand $\begin{pmatrix}\dot{u} & \dot{v}\end{pmatrix} I \begin{pmatrix}\dot{u} \\ \dot{v}\end{pmatrix}$ in terms of $\langle \mathbf{X}_u, \mathbf{X}_u \rangle, \langle \mathbf{X}_u, \mathbf{X}_v \rangle, \langle \mathbf{X}_v, \mathbf{X}_v \rangle$
4. Compare $\|\mathbf{r}'(t)\|$ and your expansion
5. Use Chapter 2 Arc-length Formula: $s = \displaystyle\int_a^b \|\mathbf{r}'(t)\|dt$

Key Takeaway. The $\underbrace{\textbf{Metric Tensor}}_{\text{first fundamental form}}$ I controls $\underbrace{\textbf{Intrinsic Invariants}}_{\text{Area / Arc-length}}$. ☐

Exercise 2.1 (Derive some familiar formula with metric tensor)**.**

1. Try $\mathbf{X}(u,v) = (u,v,0)$, then you can get $s = \displaystyle\int_a^b \sqrt{(u'(t)^2 + v'(t)^2}dt$
2. Try $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ and substitute $(u,v) = (r(\theta), \theta)$, then you can get

$$s = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}d\theta$$

3 Tangent Space & Calculations

Definition 3.1. Let S be a regular surface with parametrisation $\mathbf{X}(u, v)$, then

- The unit normal vector of S at (u, v) is

$$\mathbf{n}(u,v) = \frac{\mathbf{X}_\mathrm{u} \times \mathbf{X}_\mathrm{v}}{\|\mathbf{X}_\mathrm{u} \times \mathbf{X}_\mathrm{v}\|}$$

- The tangent space of S at $p = \mathbf{X}(u, v)$ is

$$T_pS = \{\alpha\, \mathbf{X}_u + \beta\, \mathbf{X}_v : \alpha, \beta \in \mathbb{R}\} = \{\mathbf{v} \in \mathbb{R}^3 : \langle\, \mathbf{v}, \mathbf{n}\, \rangle = 0\}$$

4 Quick Compute First Fundamental Form

Theorem 4.1. Suppose $\mathbf{X}(u, \theta) = (f(u)\cos\theta, f(u)\sin\theta, g(u))$ is a regular parametrised surface, for $\theta \in (0, 2\pi)$ and $u \in (a, b) \subset \mathbb{R}^1$, then

1. First Fundamental Form $I = \begin{pmatrix} (f'(u))^2 + (g'(u))^2 & 0 \\ 0 & (f(u))^2 \end{pmatrix}$
2. Surface Area of $\mathbf{X}(U) = 2\pi \int_a^b f(u)\sqrt{[f'(u)]^2 + [g'(u)]^2} du$

Corollary 4.2 (Sphere)**.**

- Parametrisation: $\mathbf{X}(\theta, \phi) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi), (\theta, \phi) \in (0, 2\pi) \times (0, \pi)$
- Identification: $f(\phi) = r\sin\phi$ and $g(\phi) = r\cos\phi$ and $\phi \in (0, \pi)$
- First Fundamental Form: $I = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\phi \end{pmatrix}$
- Surface Area: $A = 2\pi \int_0^\pi r^2 \sin\phi d\phi = 4\pi r^2$

Corollary 4.3 (Torus)**.**

- Parametrisation: $\mathbf{X}(\theta, \phi) = ((R + r\sin\phi)\cos\theta, (R + r\sin\phi)\sin\theta, r\cos\phi), (\theta, \phi) \in (0, 2\pi)^2$
- Identification: $f(\phi) = R + r\sin\phi$ and $g(\phi) = r\cos\phi$ and $\phi \in (0, 2\pi)$
- First Fundamental Form: $I = \begin{pmatrix} (f')^2 + (g')^2 & 0 \\ 0 & f^2 \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\sin\phi)^2 \end{pmatrix}$
- Surface Area: $A = 2\pi \int_0^{2\pi} r(R + r\sin\phi) d\phi = 4\pi^2 rR$

5 Constant Multivariable Function

Exercise 5.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_0, y_0) = 1$. Suppose $\frac{\partial}{\partial x} f = \frac{\partial}{\partial y} f = 0$ everywhere, then

$$f(x, y) = 1$$

Warm Reminder. This exercise provides a useful way to check a function is constant

6 Revision for Test 2

Warm Reminder. For "if A, then B" problems, try to think about the converse.

Exercise 6.1. Make sure you can compute $\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau$ of a space curve **correctly**

Exercise 6.2. **Revise** question type**s** of Theoretical Frenet Serret Equation problems
You may reference "Problem Set in Frenet Serret Equation"

Exercise 6.3 (Lecture Notes Chapter 2 Exercise 10)**.** Let $\mathbf{r}(t)$ be a regular parametrised curve and $\kappa(t)$ be its curvature. Prove that if $\kappa(t) = 0$ for any t, then $\mathbf{r}(t)$ is a straight line

Exercise 6.4 (Lecture Notes Chapter 3 Exercise 16)**.** Let $\mathbf{r}(t)$ be a regular parametrised space curve with $\kappa(t) > 0$ for any t. Suppose $\tau(t) = 0$ for any t, where $\tau(t)$ is the torsion at $\mathbf{r}(t)$. Prove that $\mathbf{r}(t)$ is contained in a plane

Exercise 6.5 (Lecture Notes Chapter 3 Exercise 1)**.** Prove that a regular parametrised surface in $\mathbb{R}^3$ is contained in a plane if and only if all unit normal vector $\mathbf{n}$ is constant

Exercise 6.6. Try out Problem Set on First Fundamental Form (Typed by Max Shung)

Exercise 6.7. Ask your TA for Shortlisted Problems for Test 2, if you want some challenges